Stabilization of Euler-Bernoulli beam by nonlinear boundary feedback

Francis Conrad, Michel Pierre

UNITÉ DE RECHERCHE
INRIA-LORRAINE

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P.105
78153 Le Chesnay Cedex
France
Tél.(1) 39 63 55 11

Rapports de Recherche

N° 1235

Programme 7
Calcul Scientifique,
Logiciels Numériques et Ingénierie Assistée

STABILIZATION OF EULER-BERNOULLI BEAM BY NONLINEAR BOUNDARY FEEDBACK

Francis CONRAD
Michel PIERRE

Juin 1990

* R R _ 1 2 3 5 *

STABILIZATION OF EULER-BERNOULLI BEAM BY NONLINEAR BOUNDARY FEEDBACK

Francis CONRAD *, Michel PIERRE *

Abstract

We study the damping of transversal vibrations of a system of non-homogeneous connected Euler-Bernoulli beams. Controls are forces or torques applied at one end of the system. These controls are assumed to be nonlinear functions of the observed velocities of deflection.

We show that the problem is well-posed, and that asymptotic stability is achieved when the nonlinear feedback is monotone dissipative. No assumption involving monotonicity in the bending moment or linear mass distributions is necessary.

STABILISATION D'UNE POUTRE D'EULER-BERNOULLI PAR CONTROLE FRONTIERE NON LINEAIRE

Résumé

Nous étudions l'amortissement des vibrations transversales d'un assemblage de poutres non homogènes d'Euler-Bernoulli. On contrôle par des forces ou des moments appliqués à l'une des extrémités du système. Les contrôles sont supposés être des fonctions non linéaires des vitesses de déflection observées.

Nous montrons que le problème est bien posé et qu'il y a stabilisation asymptotique quand le feedback non linéaire est monotone dissipatif. Aucune hypothèse n'est faite sur la monotonie des coefficients de densité linéique ou de rigidité en flexion.

(*) Université de Nancy I, U.R.A. CNRS 0750 and Projet NUMATH, INRIA Lorraine
Département de Mathématiques, B.P. 239, 54506 VANDOEUVRE-lès-NANCY, France.

1. INTRODUCTION

We consider an Euler-Bernoulli beam with variable mass density and flexural rigidity. The beam is clamped at one end. The other end is controlled by a point force and a point bending moment which are assumed to be nonlinear functions of the observation. By observation we mean velocity and angular velocity of the transversal deflection at the end.

Let $u(x, t)$ denote the transversal deflection at point x and time t. For a beam of unit length with flexural rigidity $a(x)$ and linear mass density $b(x)$, we get the following system of equations:

Open-loop system

$$\begin{cases} b\, u_{tt} + (a\, u_{xx})_{xx} = 0 \text{ in } (0, 1) \\ u(0, t) = u_x(0, t) = 0 \quad \text{(the beam is clamped at } x = 0) \\ -(a\, u_{xx})(1, t) = f_1(t) \quad \text{(moment)} \\ (a\, u_{xx})_x(1, t) = f_0(t) \quad \text{(force)} \\ u(x, 0) = u_0(x),\ u_t(x, 0) = u_1(x) \end{cases} \tag{1}$$

with $a \in L^\infty(0, 1)$, $b \in L^\infty(0, 1)$, $0 < m \leq a, b$. We refer to [RU] where several mathematical models for the elastic beam are presented.

Closed-loop system

Extending the case of linear feedback [CHE-DE, CHE-KRA], we assume the controls f_0 and f_1 to be nonlinear nondecreasing functions of $u_t(1, t)$ and $u_{xt}(1, t)$, vanishing at zero. This implies dissipation of the elastic energy of the system. From a practical point of view, it may be of interest to consider nonlinear feedback involving some "saturating effect" for large values of the argument, or some delay between observation and control.

Thus, we consider in this paper general feedback laws of the form:

$$f_0(t) \in \alpha(u_t(1, t))$$

$$f_1(t) \in \beta(u_{xt}(1, t))$$

where α and β are maximal monotone graphs in $\mathbb{R}^2$ such that $0 \in \alpha(0)$, $0 \in \beta(0)$. This will imply dissipativity since $\xi\, \alpha(\xi)$ and $\xi\, \beta(\xi) \subset \mathbb{R}_+$.

Therefore the system with boundary feedback control we study is the following:

$$\begin{cases} b\, u_{tt} + (a\, u_{xx})_{xx} = 0 \quad \text{in } (0, 1) \\ u(0, t) = u_x(0, t) = 0 \\ -(a\, u_{xx})(1, t) \in \beta\, (u_{xt}(1, t)) \\ (a\, u_{xx})_x(1, t) \in \alpha\, (u_t(1, t)) \end{cases} \tag{2}$$

Typically our formulation includes the case of serially connected beams, with regular (variable) mass density and flexural rigidity for each beam, strongly connected, with no force or bending moment applied at the internal nodes.

Boundary stabilization of elastic systems is presently the subject of extensive work. The case of the wave equation with nonlinear boundary feedback has been considered in [LA1], in [WA-CHE] for the asymptotic stability properties and in [ZU] for estimates.

For plates, also with nonlinear boundary feedback acting through shear forces and moments, we refer to [LA2], [LA3] for stabilization results and [LA] for estimates of the decay. The more difficult case of control by moment only has been studied in [LA4]. All these papers concern systems with constant coefficients, and, except in [WA-CHE], are based on multiplier techniques.

The case of serially connected beams has also been considered, with linear feedback acting genuinely on the force at the nodes [CHE-DE]. Exponential stability is proved in the case of nondecreasing density and nonincreasing flexural rigidity. The same result has been proved in the more difficult case of control by moment only, for a single homogeneous beam [CHE-KRA].

This brief survey is certainly far from being complete, but, to our best knowledge, the case of variable coefficients with nonlinear boundary feedback has received much less interest. In particular, most of the techniques developed so far, when applied to systems with variable coefficients, generally require strong and somewhat unnatural assumptions on the structure or on the regularity of the coefficients, this even in one space-dimension. Extra assumptions on the coefficients like monotonicity [CHE-DE] are likely to be unnecessary in order to get the decay of the energy to 0.

It is our purpose here to prove that stabilization occurs in (2) with no restriction on the functions a and b. The proof relies on the use of spectral methods and Fourier series and not on multiplier techniques.

<u>The paper is organized as follows</u>:

In Section 2, we show that Problem (2) is well-posed, in the sense that it defines a nonlinear semi-group of contractions generated by a nonlinear maximal monotone operator , on the natural energy space.

In Section 3, using the invariance principle of LASALLE [DA-SLE], combined with elementary nonharmonic Fourier analysis, we obtain the strong stability under "minimal" assumptions on the graphs α and β. In particular, stability is achieved if the system is controlled either by a shear force only or by a bending moment only.

2. WELLPOSEDNESS OF SYSTEM (2)

We set $H = L^2(0, 1)$, equipped with the norm

$$\|v\|_H = [\int_0^1 b\, v^2\, dx]^{1/2},$$

and

$$V = \{u \in H^2(0, 1)\ ;\ u(0) = u_x(0) = 0\},$$

equipped with the norm

$$\|u\|_V = [\int_0^1 a\, u_{xx}^2\, dx]^{1/2},$$

which is equivalent to the usual $H^2(0, 1)$-norm on V. The "energy space" V x H is equipped with the norm:

$$\|(u, v)\|_{VxH} = [\int_0^1 a\, u_{xx}^2\, dx + \int_0^1 b\, v^2\, dx]^{1/2}.$$

We also assume that $a \in L^\infty(0, 1)$, $b \in L^\infty(0, 1)$, $0 < m \le a, b$.

Dividing the equation in (2) by b and setting $v = u_t$, we write (2) in the form:

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} + A(u, v) \ni 0, \quad (u, v) \in V \times H \tag{3}$$

where the nonlinear operator A is defined on the space V x H by:

$$\begin{aligned} \mathrm{Dom}\,(A) = \{\ (u, v) \in V \times H;\ a\, u_{xx} \in H^2(0, 1),\ v \in V,\\ -(a\, u_{xx})(1) \in \beta(v_x(1)),\ (a\, u_{xx})_x(1) \in \alpha(v(1)\ \}\end{aligned}$$

$$\forall\ (u, v) \in \mathrm{Dom}\,(A),\quad A(u, v) = \begin{pmatrix} -v \\ \frac{1}{b}(a\, u_{xx})_{xx} \end{pmatrix}.$$

To get the wellposedness of system (2), we prove that A is a maximal monotone operator with dense domain, thus generating a strongly continuous nonlinear semi-group of contractions [PA].

Lemma 2.1. *Dom (A) is dense in V x H.*

Proof. We recall that

$$\mathrm{Dom}\,(A) = \{\ (u, v) \in V \times V;\ a\, u_{xx} \in H^2(0, 1),\ -(a\, u_{xx})(1) \in \beta(v_x(1)),\ (au_{xx})_x(1) \in \alpha(v(1))\ \}.$$

Let $E = \{ u \in V;\ a\,u_{xx} \in H^2(0,1);\ (a\,u_{xx})(1) = (a\,u_{xx})_x(1) = 0 \}$. Since $0 \in \alpha(0)$, $0 \in \beta(0)$, $E \times \mathcal{D}(0,1) \subset \mathrm{Dom}(A) \subset V \times H$. Obviously, E is dense in V, and $\mathcal{D}(0,1)$ is dense in H. Therefore $E \times \mathcal{D}(0,1)$ is dense in $V \times H$. ♦

Lemma 2.2. *A is monotone on V x H.*

Proof. Let (u, v) and $(\hat{u}, \hat{v}) \in \mathrm{Dom}(A)$. Then:

$$\left(A(u,v) - A(\hat{u},\hat{v}) \;\middle|\; \begin{matrix} u - \hat{u} \\ v - \hat{v} \end{matrix} \right)_{V\times H}$$

$$= - \int_0^1 a\ (u-\hat{u})_{xx}\,(v-\hat{v})_{xx} + \int_0^1 [a(u-\hat{u})_{xx}]_{xx}\,(v-\hat{v})$$

$$= - \int_0^1 a\ (u-\hat{u})_{xx}\,(v-\hat{v})_{xx} + [(a(u-\hat{u})_{xx})_x\,(v-\hat{v})]_0^1$$

$$- [(a(u-\hat{u})_{xx})\,(v-\hat{v})_x]_0^1 + \int_0^1 a\ (u-\hat{u})_{xx}\,(v-\hat{v})_{xx}$$

$$= (a(u-\hat{u})_{xx})_x(1)\,(v-\hat{v})(1) - (a(u-\hat{u})_{xx})(1)\,(v-\hat{v})_x(1)$$

$$\in [v(1) - \hat{v}(1)]\,[\alpha(v(1)) - \alpha(\hat{v}(1))] + [v_x(1) - \hat{v}_x(1)]\,[\beta(v_x(1)) - \beta(\hat{v}_x(1))]$$

$\subset \mathbb{R}_+$ since the graphs α and β are monotone . ♦

The maximal monotonicity of A on V x H will be established if we prove that, for any $(f, g) \in V \times H$

$$(u, v) + A(u, v) = (f, g) \tag{4}$$

admits a (unique) solution $(u, v) \in \mathrm{Dom}\,(A)$. First, we note that (4) can be rewritten as:

$$\begin{cases} u - v = f \\ v + \dfrac{1}{b}(a\,u_{xx})_{xx} = g \\ (u, v) \in \mathrm{Dom}\,(A) \end{cases} \tag{4'}$$

If $u \in V$, then $v = u - f \in V$. Therefore, it is equivalent to prove that the following problem

$$\left\{\begin{array}{l} u + \frac{1}{b}(a\, u_{xx})_{xx} = h \\ u(0) = u_x(0) = 0 \\ -(a\, u_{xx})(1) \in \beta(u_x(1) - f_x(1)) \\ (a\, u_{xx})_x(1) \in \alpha(u(1) - f(1)) \end{array}\right. \tag{5}$$

admits a solution $u \in V$, $(a\, u_{xx})_{xx} \in H$, for any $h \in H$, $f \in V$, and then set $h = f + g$, $v = u - f$. In other words, we have to prove that the following unbounded operator B defined on H by:

$$\mathrm{Dom}\,(B) = \{u \in V;\ (a\, u_{xx})_{xx} \in H;\ -(a\, u_{xx})(1) \in \beta(u_x(1) - f_x(1));\ (a\, u_{xx})_x(1) \in \alpha(u(1) - f(1))\}$$

$$\forall\, u \in \mathrm{Dom}\,(B)\quad B\, u = \frac{1}{b}(a\, u_{xx})_{xx}$$

is *maximal monotone on H*.

The monotonicity of B is easy to check: if $u, \hat{u} \in \mathrm{Dom}\,(B)$,

$$(u - \hat{u}, B\, u - B\, \hat{u})_H = \int_0^1 (u - \hat{u})\, [a\, (u - \hat{u})_{xx}]_{xx}\, dx =$$

$$\int_0^1 a\, (u - \hat{u})_{xx}^2\, dx + [(u - \hat{u})(a\, (u - \hat{u})_{xx})_x]_0^1 - [(u - \hat{u})_x\, a\, (u - \hat{u})_{xx}]_0^1 \geq$$

$$[(u - \hat{u})(a\, (u - \hat{u})_{xx})_x]\,(1) - [(u - \hat{u})_x\, a\, (u - \hat{u})_{xx}]\,(1) \in$$
$$[u(1) - \hat{u}(1)]\, [\alpha(u(1)) - \alpha(\hat{u}(1))] + [u_x(1) - \hat{u}_x(1)][\beta(u_x(1)) - \beta(\hat{u}_x(1))]\,.$$

And the last expression is a subset of $\mathbb{R}_+$ since α and β are monotone .

In order to show that (5) admits a solution, we introduce the Yosida- approximations of the graphs α and β:

$$\alpha_\lambda = \frac{I - (I + \lambda\alpha)^{-1}}{\lambda} \qquad \beta_\lambda = \frac{I - (I + \lambda\beta)^{-1}}{\lambda}\,.$$

The functions α_λ and β_λ are Lipschitz continuous and maximal monotone from $\mathbb{R}$ to $\mathbb{R}$.

We consider then the "regularization" of problem (5)

$$\left\{\begin{array}{l} u + \frac{1}{b}(a\, u_{xx})_{xx} = h \\ u(0) = u_x(0) = 0 \\ -(a\, u_{xx})(1) = \beta_\lambda(u_x(1) - f_x(1)) \\ (a\, u_{xx})_x(1) = \alpha_\lambda(u(1) - f(1)) \end{array}\right. \tag{6}$$

Lemma 2.3. *For any $h \in H, f \in V$, Problem (6) admits a unique solution $u^\lambda \in V$, such that $(a\, u^\lambda_{xx})_{xx} \in H$.*

Proof. Since α and β are maximal monotone, we have $\alpha = \partial j_1$, $\beta = \partial j_2$, where j_1 and j_2 are proper, convex, l.s.c., with Dom $(\alpha) \subset$ Dom (∂j_1) , Dom $(\beta) \subset$ Dom (∂j_2) [BRE1]. Moreover, $\alpha_\lambda = \partial j_{1\lambda}$, $\beta_\lambda = \partial j_{2\lambda}$ [BRE1] where

$$j_{1\lambda}(\xi) = \frac{\lambda}{2} |\alpha_\lambda(\xi)|^2 + j_1 [(I + \lambda\alpha)^{-1}(\xi)]$$

$$j_{2\lambda}(\xi) = \frac{\lambda}{2} |\beta_\lambda(\xi)|^2 + j_2 [(I + \lambda\beta)^{-1}(\xi)].$$

We introduce the following functional, defined for any $v \in V$:

$$\Psi_\lambda (v) = \frac{1}{2} \int_0^1 [b\, v^2 + a\, v^2_{xx} - 2\, b\, h\, v]\, dx + j_{1\lambda}(v(1) - f(1)) + j_{2\lambda}(v_x(1) - f_x(1)).$$

The functional Ψ_λ is convex, differentiable, l.s.c. and satisfies $\lim \Psi_\lambda(v) = +\infty$ as $|v|_V \to +\infty$.

Let $u^\lambda \in V$ be a minimizer of Ψ_λ. Then, for any $\phi \in V$:

$$< \Psi'_\lambda(u^\lambda), \phi >_{V'xV} = 0, \quad \text{i.e. } \forall\, \phi \in V$$

$$\int_0^1 [b\, u^\lambda\, \phi + a\, u^\lambda_{xx}\, \phi_{xx} - b\, h\, \phi] dx + \alpha_\lambda[u^\lambda(1) - f(1)]\, \phi(1) + \beta_\lambda[u^\lambda_x(1) - f_x(1)]\, \phi_x(1) = 0, \quad (7).$$

First we choose $\phi \in \mathcal{D}(0, 1)$ in (7) and obtain:

$$b\, u^\lambda + (a\, u^\lambda_{xx})_{xx} = b\, h \;\text{ in } \mathcal{D}'(0, 1),\; u^\lambda \in V.$$

Therefore $(a\, u^\lambda_{xx})_{xx} \in H$; the above equation has a sense in $L^2(0, 1)$ and we can divide it by b to recover the first equation of (6).

Next we choose $\phi \in V$ in (7) and integrate by parts (this is valid since $(a\, u^\lambda_{xx})_{xx} \in H$):

$$\int_0^1 [b\, u^\lambda + (a\, u^\lambda_{xx})_{xx} - b\, h]\, \phi\, dx + [a\, u^\lambda_{xx}\, \phi_x - (a\, u^\lambda_{xx})_x\, \phi]_0^1$$

$$+ \alpha_\lambda[u^\lambda(1) - f(1)]\, \phi(1) + \beta_\lambda[u^\lambda_x(1) - f_x(1)]\, \phi_x(1) = 0,$$

which implies

$$(a\, u^\lambda_{xx})(1) = -\beta_\lambda(u^\lambda_x(1) - f_x(1)) \quad \text{and} \quad (a\, u^\lambda_{xx})_x(1) = \alpha_\lambda(u^\lambda(1) - f(1)).$$

This completes the proof of Lemma 2.3. ♦

Remark. It is also possible to get existence for Problem (6) by linearizing (6) and using a fixed point argument.

Now we return to Problem (5) and prove the maximal monotonicity of B.

Lemma 2.4. *For any $h \in H, f \in V$, Problem (5) admits a unique solution $u \in V$, such that $a\, u_{xx} \in H^2(0, 1)$. Moreover, there exists a constant $C > 0$ such that:*

$$|u|_V^2 + |a\, u_{xx}|_{H^2}^2 \le C\, [\, |h|_H^2 + |f|_V^2].$$

Proof. We first establish estimates for the regularized problem (6), then pass to the limit as $\lambda \to 0$. To do so, we essentially argue as in [BRE2] where similar nonlinear boundary conditions are considered.

Estimates for Problem (6) . Let u^λ be the solution of (6).

a) Multiply the equation in (6) by $b\, u^\lambda$ and integrate:

$$\int_0^1 [b\, (u^\lambda)^2 + a\, (u^\lambda_{xx})^2]\, dx = \int_0^1 b\, h\, u^\lambda\, dx - u^\lambda(1)\, \alpha_\lambda[u^\lambda(1) - f(1)] - u^\lambda_x(1) \beta_\lambda[u^\lambda_x(1) - f_x(1)]\, .$$

Since $\xi\, \alpha_\lambda(\xi)$ and $\xi\, \beta_\lambda(\xi)$ are nonnegative, we have

$$u^\lambda(1)\, \alpha_\lambda[u^\lambda(1) - f(1)] \ge f(1)\, \alpha_\lambda[u^\lambda(1) - f(1)]$$

$$u^\lambda_x(1)\, \beta_\lambda[u^\lambda_x(1) - f_x(1)] \ge f_x(1)\, \beta_\lambda[u^\lambda_x(1) - f_x(1)]$$

therefore

$$\int_0^1 [b\, (u^\lambda)^2 + a\, (u^\lambda_{xx})^2]\, dx \le \int_0^1 b\, h\, u^\lambda\, dx + |f(1)\, (a\, u^\lambda_{xx})_x(1)| + |f_x(1)\, a\, u^\lambda_{xx}(1)|.$$

b) Multiply the equation in (6) by $(a\, u^\lambda_{xx})_{xx}$ and integrate:

$$\int_0^1 [a\, (u^\lambda_{xx})^2 + \frac{1}{b}\, ((a\, u^\lambda_{xx})_{xx})^2]\, dx = \int_0^1 h\, (a\, u^\lambda_{xx})_{xx}\, dx$$

$$- u^\lambda(1)\, \alpha_\lambda[u^\lambda(1) - f(1)] - u^\lambda_x(1)\, \beta_\lambda[u^\lambda_x(1) - f_x(1)].$$

Using then the same trick as in a) for the last two terms, we get the estimate:

$$\int_0^1 [a\,(u^{\lambda}_{xx})^2 + \frac{1}{b}\,((a\,u^{\lambda}_{xx})_{xx})^2]\,dx \;\le\; \int_0^1 h\,(a\,u^{\lambda}_{xx})_{xx}\,dx \;+\; |f(1)\,(a\,u^{\lambda}_{xx})_x(1)| \;+\; |f_x(1)\,a\,u^{\lambda}_{xx}(1)|$$

c) Summing the estimates obtained in a) and b), we get:

$$\int_0^1 b\,(u^{\lambda})^2\,dx + 2\int_0^1 a\,(u^{\lambda}_{xx})^2\,dx + \int_0^1 \frac{1}{b}\,[(a\,u^{\lambda}_{xx})_{xx}]^2\,dx \;\le$$

$$\int_0^1 b\,h\,u^{\lambda}\,dx \;+\; \int_0^1 h\,(a\,u^{\lambda}_{xx})_{xx}\,dx \;+ 2\,|f(1)\,(a\,u^{\lambda}_{xx})_x(1)| \;+\; 2\,|f_x(1)\,a\,u^{\lambda}_{xx}(1)|\,. \qquad (8)$$

d) Now we use the following simple inequalities (C denotes various constants):

. $|f(1)| \le C\,|f|_V;\quad |f_x(1)| \le C\,|f|_V$

. $|(a\,u^{\lambda}_{xx}(1)|^2 \le C\,[\,|a\,u^{\lambda}_{xx}|^2_H + |(a\,u^{\lambda}_{xx})_{xx}|^2_H\,]$

$$\le C\,[\int_0^1 a\,(u^{\lambda}_{xx})^2\,dx \;+\; \int_0^1 \frac{1}{b}\,[(a\,u^{\lambda}_{xx})_{xx}]^2\,dx$$

. $|(a\,u^{1}_{xx})_x(1)|^2 \;\le\; C\,[\int_0^1 a\,(u^{\lambda}_{xx})^2\,dx \;+\; \int_0^1 \frac{1}{b}\,[(a\,u^{\lambda}_{xx})_{xx}]^2\,dx\,.$

We plug these inequalities into the estimate (8) and apply Young's inequality to all terms of the R.H.S. of (8) with suitable coefficients (small enough relatively to the uncontrolled terms: u^{λ}, $(a\,u^{\lambda}_{xx})_{xx}$, $(a\,u^{\lambda}_{xx})_x(1)$, $(a\,u^{\lambda}_{xx})(1)$). We obtain (using also the positivity of b):

$$\int_0^1 b\,(u^{\lambda})^2\,dx + \int_0^1 a\,(u^{\lambda}_{xx})^2\,dx + \int_0^1 [(a\,u^{\lambda}_{xx})_{xx}]^2\,dx \;\le\; C\;[\,|h|^2_H + |f|^2_V] \quad . \qquad (9)$$

Passing to the limit as $\lambda \to 0$. From (9), we deduce that, as $\lambda \to 0$ (and, as usual, up to a subsequence):

. $u^{\lambda} \to u$, weakly in V, strongly in $C^1([0, 1])$

. $a\,u^{\lambda}_{xx} \to v$, weakly in $H^2(0, 1)$, strongly in $C^1([0, 1])$

Thus $u(0) = u_x(0) = 0$, and $v = a\,u_{xx}$ in $L^2(0, 1)$. We deduce also from (9) that $(a\,u^\lambda_{xx})_{xx} \to v_{xx}$, weakly in $L^2(0, 1)$. Therefore $u + \frac{1}{b}(a\,u_{xx})_{xx} = h$ a.e. in $L^2(0, 1)$.

Finally, we have to study the limit, as $\lambda \to 0$, of the boundary conditions:

$$-(a\,u^\lambda_{xx})(1) = \beta_\lambda(u^\lambda_x(1) - f_x(1))\,;\ (a\,u^\lambda_{xx})_x(1) = \alpha_\lambda(u^\lambda(1) - f(1)).$$

For instance for the second equation, we set

$$x_\lambda = (a\,u^\lambda_{xx})_x(1),\ y_\lambda = (I + \lambda\alpha)^{-1}\,(u^\lambda(1) - f(1)).$$

First, since $a\,u^\lambda_{xx} \to a\,u_{xx}$ in $C^1([0,1])$, $\lim x_\lambda = (a\,u_{xx})_x(1)$ as $\lambda \to 0$. Then

$$|(a\,u^\lambda_{xx})_x(1)| = |\alpha_\lambda[u^\lambda(1) - f(1)]| = \frac{|u^\lambda(1) - f(1) - (I + \lambda\alpha)^{-1}\,(u^\lambda(1) - f(1))|}{\lambda}$$
$$= \frac{|u^\lambda(1) - f(1) - y_\lambda|}{\lambda}$$

is bounded; thus $\lim y_\lambda = u(1) - f(1)$ as $\lambda \to 0$. But

$$x_\lambda = \alpha_\lambda(u^\lambda(1) - f(1)) \in \alpha((I + \lambda\alpha)^{-1}\,(u^\lambda(1) - f(1))) = \alpha(y_\lambda)\ \text{[BRE1]}.$$

Summarizing the preceding results, we have:

$$x_\lambda \in \alpha(y_\lambda),\ \lim x_\lambda = (a\,u_{xx})_x(1) \text{ as } \lambda \to 0,\ \lim y_\lambda = u(1) - f(1) \text{ as } \lambda \to 0.$$

By [BRE1, Prop. 2.2], we obtain $(a\,u_{xx})_x(1) \in \alpha(u(1) - f(1))$.

The proof for the boundary condition involving β is similar.

We conclude that the limit u is a solution of (5), with $u \in V$, $a\,u_{xx} \in H^2(0, 1)$, and in particular $(a\,u_{xx})_{xx} \in H$.

The estimate of Lemma 2.4 is an immediate consequence of (9). The proof is now complete. ♦

Proposition 2.5. *A is a maximal monotone operator with dense domain. Moreover, for all (f,g) in V x H, $(u, v) = (I + A)^{-1}(f, g)$ satisfies:*

$$|u|^2_V + |v|^2_V + |a\,u_{xx}|^2_{H^2} \le C\,[\,|f|^2_V + |g|^2_H\,].$$

Proof. This result immediately follows from Lemmas 2.1, 2.2, 2.4, and the fact that (4) and (5) are equivalent provided we set $h = f + g$, $v = u - f$ (see (4)' and (5)). ♦

Now the evolution problem (2) makes sense in a standard way.

Theorem 2.6. *(i) For any initial data (u_0, u_1) in $V \times H$, Problem (3) admits a unique "mild solution". In fact, (3) defines a nonlinear semi-group of contractions on $V \times H$.*
(ii) If $(u_0, u_1) \in Dom(A)$, then u is a strong solution of (2) or (3), i.e.:

$(u, u_t) \in Dom(A)$ *a.e.(t);*

$t \in (0, +\infty) \to (u, u_t) \in V \times H$ *is (a.e.) differentiable w.r.t. t,*

and (2) or (3) are satisfied a.e. in t.

Proof. For Problem (3), (i) and (ii) are direct consequences of Proposition 2.5 by standard results [BRE1, PA]. We just have to notice that for (ii), $u_{tt} \in H$, $\frac{1}{b}(a\, u_{xx})_{xx} \in H$, so that we can multiply equation (3) by b to get (2). Let us recall that Dom (A) =

$$\{(u, v) \in V \times H;\ (a\, u_{xx}) \in H^2(0, 1);\ v \in V;\ -(a\, u_{xx})(1) \in \beta(v_x(1));\ (au_{xx})_x(1) \in \alpha(v(1)\}.\ \blacklozenge$$

Remark. As mentioned in the introduction, Theorem 2.6 is relevant for a system of serially connected beams with variable (piecewise regular) mass densities and flexural rigidities, with no force or moment acting on the internal nodes. The system is clamped at the left end, and control is exerted at the right end.

We now have the material to study the asymptotic behaviour of Problems (2) or (3).

3. ASYMPTOTIC BEHAVIOUR

We prove the strong asymptotic stability of the solutions of (3) in V x H, using the *invariance principle of LASALLE* [DA-SLE, SLE].

Let $(u, u_t) = S(t)(u_0, u_1)$ be the solution of (3) with initial conditions $(u_0, u_1) \in V \times H$. We define the elastic energy of the solution (u, u_t) by

$$E(t) = \frac{1}{2} \|(u(t), u_t(t))\|^2_{V\times H} = \frac{1}{2} \int_0^1 [b\, u_t^2(x, t) + a\, u_{xx}^2(x, t)]\, dx.$$

Lemma 3.1. (i) E is nonincreasing w.r.t. t

(ii) if $(u_0, u_1) \in$ Dom (A), then $E(t) - E(0) \in -\int_0^t [\alpha(u_t(1, s))u_t(1, s) + \beta(u_{xt}(1, s))u_{xt}(1, s)]\, ds$

Proof. (ii) Let $(u_0, u_1) \in$ Dom (A); by Theorem 2.6 and standard results [BRE1, PA], the mapping $t \in (0, +\infty) \rightarrow (u, u_t) \in V \times H$ is Lipschitz continuous in V x H, and a.e. differentiable. Thus we apply the dominated convergence theorem and differentiate E(t):

$$\frac{dE}{dt}(t) = \int_0^1 [b\, u_t(x, t)\, u_{tt}(x, t) + a\, u_{xx}(x, t)\, u_{xxt}(x, t)]\, dx.$$

Since $(u, u_t) \in$ Dom (A), $u_t \in V$ and $a\, u_{xx} \in H^2(0, 1)$. We can integrate the first term in the integral by parts, replacing $b\, u_{tt}$ by $-(a\, u_{xx})_{xx} \in L^2(0, 1)$:

$$\begin{aligned}\frac{dE}{dt}(t) &= -[u_t(x, t)\, (a\, u_{xx})_x(x, t)]_0^1 + [u_{xt}(x, t)\, (a\, u_{xx})(x, t)]_0^1 \\ &= -\, u_t(1, t)\, (a\, u_{xx})_x(1, t) + u_{xt}(1, t)\, (a\, u_{xx})(1, t) \\ &\in -\, \alpha(u_t(1, t))\, u_t(1, t) - \beta(u_{xt}(1, t))\, u_{xt}(1, t) \subset \mathbb{R}_-.\end{aligned}$$

Integrating from 0 to t gives the energy formula (ii) of Lemma 3.1.

(i) In particular, E(t) is nonincreasing.

By the density of Dom (A) in V x H, and the contraction property of the semi-group, this last result is also valid if $(u_0, u_1) \in V \times H$. Thus (i) is also established. ♦

Lemma 3.2. The resolvent operator $(I + \lambda A)^{-1}$ is compact from V x H to V x H and $0 \in \mathcal{R}(A)$.

Proof. Since $0 \in \alpha(0)$ and $0 \in \beta(0)$, we have $0 \in$ Dom (A) and A(0) = 0.

By the estimate of Proposition 2.5, the compactness of the resolvent will be true if we show that

$\{(u, v) \in V \times V;\ a\, u_{xx} \in H^2(0, 1)\}$ is compactly imbedded in V x H.

Since V is compactly imbedded in H, it is enough to prove that if, for a sequence u^n bounded in V, au^n_{xx} is also bounded in $H^2(0, 1)$, then u^n converges strongly in V.This is obvious since $a \geq m > 0$.

♦

It follows from Lemma 3.2 and standard results [DA-SLE, PA] that, for any $(u_0, u_1) \in V \times H$, the trajectories $\{S(t)(u_0, u_1), t \geq 0\}$ are precompact in V x H, and that the ω-limit set $\omega(u_0, u_1)$ is a non-empty, invariant, connected, compact set such that lim dist $((u(t),u_t(t)),\ \omega(u_0, u_1)) = 0$ as $t \nearrow \infty$.

By Lemma 3.1, $\frac{1}{2}\|(u, v)\|^2_{V\times H}$ is a Lyapunov function on V x H. We have now all the material to apply the *invariance principle of LASALLE*, in order to get the asymptotic stability for (3).

Let us sketch how the method works in our case.

In order to prove that $(u, u_t) \to 0$ in V x H as $t \nearrow \infty$, it is enough to prove that $\omega(u_0, u_1) = \{0\}$, for any $(u_0, u_1) \in V \times H$. By the density of Dom (A) in V x H, and the contraction property of the semi-group S(t), it is enough to prove that $\omega(u_0, u_1) = \{0\}$, for any $(u_0, u_1) \in$ Dom (A).

Consider thus $(u_0, u_1) \in$ Dom (A). Then , since A S(t) (u_0, u_1) remains bounded in VxH uniformly in t, $\omega(u_0, u_1) \subset$ Dom (A) [BRE1].

Choose $(w_0, w_1) \in \omega(u_0, u_1)$, and set $(w,w_t)(t) = S(t)(w_0, w_1) \in \omega(u_0, u_1) \subset$ Dom (A). By the above remarks, the asymptotic stability will be true if we prove that $(w_0, w_1) = 0$, or else, $(w,w_t)(t) = 0$, for any $t \geq 0$.

Now, we apply the invariance principle [DA-SLE, SLE]:

$$E(t) = \frac{1}{2}\|S(t)(w_0, w_1)\|^2_{V\times H} = \frac{1}{2}\int_0^1 [b\, w_t^2(x, t) + a\, w_{xx}^2(x, t)]\, dx \quad \underline{\text{is constant w.r.t. t.}}$$

By the energy formula (ii) of Lemma 3.1, applied to (w_0, w_1), we deduce that

$$0 \in \int_0^t [\alpha(w_t(1, s))w_t(1, s) + \beta(w_{xt}(1, s))w_{xt}(1, s)]\, ds$$

which, since $\xi\, \alpha(\xi)$ and $\xi\, \beta(\xi) \subset \mathbb{R}_+$, implies that

$$0 \in \alpha(w_t(1, t))w_t(1, t), \quad 0 \in \beta(w_{xt}(1, t))w_{xt}(1, t) \quad \text{a.e. } t.$$

Summarizing the whole discussion above, we have shown that the strong asymptotic stability of (3) will be a consequence of the following "uniqueness" result:

Let w be a solution of the following "overdetermined" system (10), (11):

$$\begin{cases} b\, w_{tt} + (a\, w_{xx})_{xx} = 0 \quad \text{in } (0, 1) \\ w(0, t) = w_x(0, t) = 0 \\ -(a\, w_{xx})(1, t) \in \beta\, (w_{xt}(1, t)) \\ (a\, w_{xx})_x(1, t) \in \alpha\, (w_t(1, t)) \\ (w_0, w_1) \in \omega(u_0, u_1) \subset \text{Dom (A)} \end{cases} \tag{10}$$

$$\begin{cases} 0 \in \alpha(w_t(1, t))w_t(1, t) \\ 0 \in \beta(w_{xt}(1, t))w_{xt}(1, t) \end{cases} \tag{11}$$

then necessarily $(w, w_t) \equiv 0$.

Now we prove this uniqueness result under weak assumptions on the graphs α and β. In fact, we give two results, one in the case when the force is genuinely active, and one in the case when the moment is genuinely active.

We obtain the results by expanding the solution of (10), (11) in Fourier series. We could also use a multiplier technique as in [LA1]. But, it seems to require monotonicity assumptions on a and b. This is not the case here. Thus it appears that, at least in the case of dimension one in space, the spectral method is preferable.

Some definitions and notations.

We introduce the operator B_0 on H, which is in fact the operator B as defined in Section 2, in the specific case $\alpha = \beta = 0$:

$\mathrm{Dom}(B_0) = \{u \in V;\ a\, u_{xx} \in H^2(0, 1);\ a\, u_{xx}(1) = (a\, u_{xx})_x(1) = 0\}$,

which is dense in H, and,

$\forall\, u \in \mathrm{Dom}(B_0),\ B_0 u = \frac{1}{b}(a\, u_{xx})_{xx}$.

The operator B_0 is self-adjoint, and by Lemma 2.4, $\mathcal{R}(I + B_0) = H$ and $(I + B_0)^{-1}$ is compact.

Let $0 < \lambda_1 \le \lambda_2 \le \dots \le \lambda_n$.. be the sequence of eigenvalues of B_0 and ϕ_n associated eigenfunctions. In other words, (λ_n, ϕ_n) satisfy the following system:

$$\begin{cases} (a\, \phi_{xx})_{xx} = \lambda\, b\, \phi \ \text{ in } (0, 1) \\ \phi(0) = \phi_x(0) = 0 \\ (a\, \phi_{xx})(1) = (a\, \phi_{xx})_x(1) = 0 \end{cases} \tag{12}$$

We assume the ϕ_n are normalized by $\|\phi_n\|_H^2 = \int_0^1 b\, \phi_n^2\, dx = 1 = \lambda_n^{-1} \int_0^1 a\, \phi_{nxx}^2\, dx$. We introduce also the operator A_0 on V x H, which is in fact the operator A as defined in Section 2, in the specific case $\alpha = \beta = 0$:

$\mathrm{Dom}(A_0) = \{ (u, v) \in V \times V;\ a\, u_{xx} \in H^2(0, 1);\ -(a\, u_{xx})(1) = (a u_{xx})_x(1) = 0 \}$,

which is dense in V x H, and,

$$\forall\ (u, v) \in \mathrm{Dom}\,(A_0): A_0\,(u, v) = \begin{pmatrix} -v \\ \frac{1}{b}(a\, u_{xx})_{xx} \end{pmatrix}.$$

The operator A_0 is skew-adjoint, and, by Proposition 2.5 and Lemma 3.2, $R(I + A_0) = V \times H$, and $(I + A_0)^{-1}$ is compact.

Let μ_n be the sequence of eigenvalues of A_0 and ψ_n the associated eigenfunctions, normalized so that $\|\psi_n\|^2_{V\times H} = 1$.

It is easy to see that $\mu_n = \varepsilon\, i \sqrt{\lambda_n}$ and we can choose $\psi_n = \dfrac{1}{\sqrt{2\,\lambda_n}} \begin{pmatrix} \phi_n \\ -\varepsilon\, i \sqrt{\lambda_n}\ \phi_n \end{pmatrix}$,

where $\varepsilon = \pm 1$, and $n = 1, 2, \ldots$

We set $\mu_n = i\sqrt{\lambda_n}$, $\psi_n = \dfrac{1}{\sqrt{2\,\lambda_n}} \begin{pmatrix} \phi_n \\ -\, i \sqrt{\lambda_n}\ \phi_n \end{pmatrix}$ for $n = 1, 2, \ldots$

and $\mu_{-n} = -\, i\sqrt{\lambda_n}$, $\psi_{-n} = \dfrac{1}{\sqrt{2\,\lambda_n}} \begin{pmatrix} \phi_n \\ i \sqrt{\lambda_n}\ \phi_n \end{pmatrix}$ for $n = 1, 2, \ldots$ ($\mu_0 = 0$, $\psi_0 = 0$).

Theorem 3.3. *Assume*
(i) $\alpha(0) = \{0\}$ *and* $\forall\, \xi \neq 0,\ 0 \notin \alpha(\xi)$
(ii) $\beta(0) = \{0\}$ *and* $\forall\, \xi,\ \ 0 \in \beta(\xi) \ \Rightarrow\ \beta(\xi) = \{0\}$.
Then, $\forall\ (u_0, u_1) \in V \times H,\ S(t)(u_0, u_1) \rightarrow 0$ *in* $V \times H$ *as* $t \nearrow \infty$.

Remark. Assumptions (i) and (ii) imply that the graphs are single-valued at 0 (no vertical part at 0); (i) implies also that the graph α is not "flat" at 0. But β may be "flat". Therefore, Theorem 3.3 applies in the particular case of feedback control by a force only ($\beta = 0$).

Theorem 3.4. *Assume*
(i) $\beta(0) = \{0\}$ *and* $\forall\, \xi \neq 0,\ 0 \notin \beta(\xi)$
(ii) $\alpha(0) = \{0\}$ *and* $\forall\, \xi,\ \ 0 \in \alpha(\xi) \ \Rightarrow\ \alpha(\xi) = \{0\}$.
Then, $\forall\ (u_0, u_1) \in V \times H,\ S(t)(u_0, u_1) \rightarrow 0$ *in* $V \times H$ *as* $t \nearrow \infty$.

Remark. Theorem 3.4 applies in the particular case of feedback control by a moment only ($\alpha = 0$).

Proof of Theorem 3.3. We solve system (10) - (11) and prove that $(w, w_t) \equiv 0$.

First, consider (11). Assumption (i) implies $w_t(1, t) = 0$ and $\alpha(w_t(1, t)) = \{0\}$, a.e. (t). Assumption (ii) implies $\beta(w_{xt}(1, t)) = \{0\}$, a.e. (t).

Therefore, system (10) reduces to

$$\begin{cases} b\, w_{tt} + (a\, w_{xx})_{xx} = 0 \text{ in } (0, 1) \\ w(0, t) = w_x(0, t) = 0 \\ -(a\, w_{xx})(1, t) = 0 \\ (a\, w_{xx})_x(1, t) = 0 \\ (w_0, w_1) \in \mathrm{Dom}\,(A) \end{cases} \tag{13}$$

or, in condensed form,

$$\frac{d}{dt}\begin{pmatrix} w \\ w_t \end{pmatrix} + A_0\,(w, w_t) \ni 0 \tag{13'}$$

with the subsidiary equation induced by (11)

$$w_t(1, t) = 0 \quad . \tag{14}$$

The solution of the linear evolution equation (13) is obtained by spectral analysis:

$$\begin{pmatrix} w \\ w_t \end{pmatrix}(t) = e^{-A_0 t} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \sum_{n \in \mathbb{Z}} c_n\, e^{-\mu_n t}\, \psi_n$$

where $c_n = \left(\begin{matrix} w_0 \\ w_1 \end{matrix} \middle| \psi_n \right)_{V \times H}$ (for the complexified scalar product on $V \times H$) and where the series converges in $V \times H$ uniformly in t. One finds that

$$c_n = a_n^* + i\, b_n^* \text{ if } n > 0, \quad c_n = a_{-n}^* - i\, b_{-n}^* \text{ if } n < 0,$$

where

$$a_n^* = \frac{1}{\sqrt{2\,\lambda_n}} \int_0^1 a(x)\, w_{0xx}(x)\, \phi_{nxx}(x)\, dx\ , \quad n = 1, 2, \ldots$$

$$b_n^* = \frac{1}{\sqrt{2}} \int_0^1 b(x)\, w_1(x)\, \phi_n(x)\, dx\ , \quad n = 1, 2, \ldots$$

After an easy computation, we get

$$w(t) = \sum_{n=1}^{n=\infty} [a_n \cos(\sqrt{\lambda_n}\ t) + b_n \sin(\sqrt{\lambda_n}\ t)]\ \frac{\phi_n}{\sqrt{\lambda_n}} \tag{15}$$

$$w_t(t) = \sum_{n=1}^{n=\infty} [b_n \cos(\sqrt{\lambda_n}\ t) - a_n \sin(\sqrt{\lambda_n}\ t)]\ \phi_n \tag{16}$$

where

$$a_n = \sqrt{2}\, a_n^* = \frac{1}{\sqrt{\lambda_n}} \int_0^1 a(x)\, w_{0xx}(x)\, \phi_{nxx}(x)\, dx$$

$$b_n = \sqrt{2}\, b_n^* = \int_0^1 b(x)\, w_1(x)\, \phi_n(x)\, dx\ .$$

The series given by (15) (resp. (16)) converges in V (resp. H). In fact, because of (13') and the right continuity of $t \to A\,(w, w_t)$ from $[0, \infty)$ into V x H, $(w_0, w_1) \in \mathrm{Dom}\,(A_0)$. Therefore these convergences can be improved. Indeed

$$w_1 = \sum_{n=1}^{n=\infty} b_n\, \phi_n \in V, \text{ thus } \sum_{n=1}^{n=\infty} \lambda_n\, b_n^2 < \infty \text{ since } |\phi_n|_V^2 = \lambda_n \text{ and}$$

$$b_n \sqrt{\lambda_n} = < w_1, \frac{\phi_n}{\sqrt{\lambda_n}} >_{VxV}.$$

We conclude that $b_n \sqrt{\lambda_n} \in l^2(\mathbb{N})$ and that the first series defining $w_t(t)$ in (16) converges in V uniformly in t.

Similarly, $w_0 = \sum_{n=1}^{n=\infty} a_n \frac{\phi_n}{\sqrt{\lambda_n}} \in V$, thus $a_n \in l^2(\mathbb{N})$. Moreover, since $(w_0, w_1) \in$ Dom (A_0), $(a\, w_{0xx})_{xx}$ is in $L^2(0, 1)$ and $a_n \sqrt{\lambda_n} = < (a\, w_{0xx})_{xx}, \phi_n >_{L^2xL^2}$.

Therefore $\displaystyle\sum_{n=1}^{n=\infty} \lambda_n\, a_n^2\, \|\phi_n\|_H^2 = \sum_{n=1}^{n=\infty} \lambda_n\, a_n^2 < \infty.$

We conclude as above that $a_n \sqrt{\lambda_n} \in l^2(\mathbb{N})$ and that the second series in (16) converges in V uniformly in t.

Therefore the series in (16) converges in V uniformly in t. By continuity of the trace operator

$v \to v(1)$ on V, the subsidiary equation (14) $w_t(1,t) = 0$ rewrites as

$$\sum_{n=1}^{n=\infty} [b_n \phi_n(1) \cos(\sqrt{\lambda_n}\ t) - a_n \phi_n(1) \sin(\sqrt{\lambda_n}\ t)] = 0 \text{ a.e. } t > 0 \qquad (17)$$

where above real valued series converges uniformly in t (a priori not absolutely).

Now, we have to deduce from (17) that the coefficients a_n and b_n are identically equal to 0. This will be a consequence of the following proposition whose proof will be given in the appendix.

Proposition 3.5. *Let λ_n be an eigenvalue of Problem (12), ϕ_n an associated eigenfunction. Then*
(i) λ_n is simple
(ii) $\phi_n(1) \neq 0$
(iii) $\phi_{nx}(1) \neq 0$

According to Proposition 3.5 (i), all the numbers $\sqrt{\lambda_n}$ appearing in (17) are distinct. It is a wellknown fact that the uniform convergence of the series (17) implies that $b_n \phi_n(1)$ and $-a_n \phi_n(1)$ are the Fourier coefficients of the uniformly almost periodic function 0. Thus

$$a_n \phi_n(1) = b_n \phi_n(1) = 0.$$

For the sake of completeness, we prove directly that result here.

Let $P_N(t) = \sum_{n=1}^{n=N} [b_n \phi_n(1) \cos(\sqrt{\lambda_n}\ t) - a_n \phi_n(1) \sin(\sqrt{\lambda_n}\ t)]$

Let $p \in \mathbb{N}$ be fixed, $\varepsilon > 0$ be arbitrary, and choose $N = N_\varepsilon \geq p$ large enough so that $|P_N(t)| \leq \varepsilon$, $\forall\, t \geq 0$. Then

$$\frac{1}{T}\int_0^T P_N(t) \cos(\sqrt{\lambda_p}\ t)\, dt =$$

$$\frac{1}{2T} \sum_{n=1}^{n=N} b_n \phi_n(1) \int_0^T [\cos((\sqrt{\lambda_n} + \sqrt{\lambda_p})t) + \cos((\sqrt{\lambda_n} - \sqrt{\lambda_p})t)]\, dt$$

$$-\frac{1}{2T} \sum_{n=1}^{n=N} a_n \phi_n(1) \int_0^T [\sin((\sqrt{\lambda_n} + \sqrt{\lambda_p})t) + \sin((\sqrt{\lambda_n} - \sqrt{\lambda_p})t)]\, dt$$

Now pass to the limit in the right hand side as $T \nearrow \infty$. For $n \neq p$ all the terms go to zero. For $n = p$ the limit of the sum is just $\frac{1}{2} b_p \phi_p(1)$ (the three other terms goe to zero, or cancel).

On the other hand, $\left|\frac{1}{T}\int_0^T P_N(t)\cos(\sqrt{\lambda_p}\,t)\,dt\right| \le \varepsilon$. Thus $|b_p\,\phi_p\,(1)| \le 2\,\varepsilon$, for any $\varepsilon > 0$.

In a similar way, one proves that $|a_p\,\phi_p\,(1)| \le 2\,\varepsilon$, for any $\varepsilon > 0$. Thus $a_p\,\phi_p\,(1) = b_p\,\phi_p\,(1) = 0$. Thanks to the observability condition (ii) in Proposition 3.5, we deduce next that

$$a_p = b_p = 0 \text{ for any } p \ge 1, \text{ which implies } (w, w_t) = 0\,.$$

Therefore, the proof of Theorem 3.3 is complete. ♦

Proof of Theorem 3.4.

As in Theorem 3.3, we consider system (10)-(11) and prove that $(w, w_t) \equiv 0$.

First, consider (11). Assumption (i) implies $w_{xt}(1, t) = 0$ and $\beta(w_{xt}(1, t)) = \{0\}$, a.e. (t). Assumption (ii) implies $\alpha(w_t(1, t)) = \{0\}$, a.e. (t). Again, system (10) reduces to

$$\begin{cases} b\, w_{tt} + (a\, w_{xx})_{xx} = 0 \text{ in } (0, 1) \\ w(0, t) = w_x(0, t) = 0 \\ -(a\, w_{xx})(1, t) = 0 \\ (a\, w_{xx})_x(1, t) = 0 \\ (w_0\,, w_1) \in \mathrm{Dom}\,(A) \end{cases} \tag{18}$$

with the subsidiary equation implied by (11)

$$w_{xt}(1, t) = 0 \tag{19}$$

The solution of Equation (18) is obtained by the same Fourier expansion as before:

$$w(t) = \sum_{n=1}^{n=\infty} [a_n \cos(\sqrt{\lambda_n}\;t) + b_n \sin(\sqrt{\lambda_n}\;t)]\;\frac{\phi_n}{\sqrt{\lambda_n}}$$

$$w_t(t) = \sum_{n=1}^{n=\infty} [b_n \cos(\sqrt{\lambda_n}\;t) - a_n \sin(\sqrt{\lambda_n}\;t)]\;\phi_n,$$

but now, the subsidiary Equation (19) rewrites as

$$\sum_{n=1}^{n=\infty} [b_n\,\phi_{nx}(1) \cos(\sqrt{\lambda_n}\;t) - a_n\,\phi_{nx}(1) \sin(\sqrt{\lambda_n}\;t)] = 0 \text{ a.e. } t > 0 \tag{20}$$

where, by continuity of the trace operator $v \rightarrow v_x(1)$ on V, the above real valued series converges uniformly in t (a priori not absolutely). We conclude as before from (20) and Proposition 3.5 (i) that $a_n\, \phi_{nx}(1)$ and $b_n\, \phi_{nx}(1) = 0$ for any $n \geq 1$. Thanks to the observability condition (iii) in Proposition 3.5, we deduce next that $a_n = b_n = 0$ for any $n \geq 1$, which implies $(w, w_t) = 0$.

Therefore the proof of Theorem 4.4. is also complete. ♦

Remark 1. A similar procedure has been used in [SLE] to obtain the asymptotic stability for a beam with dynamic boundary feedback.

Remark 2. If a and b are constant coefficients, it is possible to prove the "uniqueness" of (10)-(11) in a more refined way: if $w_t(1, t)$ (resp. $w_{xt}(1, t)$) $= 0$ on $(0, T)$ for some $T > 0$, then $(w, w_t) \equiv 0$. The proof, based on an inequality for lacunary Fourier series [BA-SLE], requires an asymptotic gap for the eigenvalues $\sqrt{\lambda_n}$ (this gap is even infinite in our case so that T can be arbitrarily small). Here we do not need such a precise assumption (simplicity of the eigenvalues is sufficient) because we use the property that <u>the observation is 0 on $(0, \infty)$</u>. In fact, we only need the observation <u>for large time</u>.

Remark 3. The method developed in this paper can be adapted (in a slightly more simple way) to the case of boundary stabilization of the wave equation by nonlinear feedback:

$$\begin{cases} b\, u_{tt} - (a\, u_x)_x = 0 \\ u(0, t) = 0 \\ -(a\, u_x)(1, t) \in \alpha\, (u_t(1, t)) \end{cases}$$

where the energy space is V x H, with $V = \{v \in H^1(0, 1);\ v(0) = 0\}$.

When a and b are constants, this problem has been studied in [WA-CHE] in the case of space dimension 1, in [LA1], [ZU] in the general case.

APPENDIX

Proof of Proposition 3.5.

Preliminary remarks. We recall the eigenvalue problem (12) which was considered in Section 3:

$$\begin{cases} (a\,\phi_{xx})_{xx} = \lambda\, b\, \phi \quad \text{in } (0,1) \\ \phi(0) = \phi_x(0) = 0 \\ (a\,\phi_{xx})(1) = (a\,\phi_{xx})_x(1) = 0 \end{cases} \tag{A_1}$$

Let $\lambda_n > 0$ be the sequence of eigenvalues of (A_1) and let ϕ_n be an associated sequence of eigenfunctions, normalized by $\|\phi_n\|_H^2 = \int_0^1 b\,\phi_n^2\,dx = 1$.

Results on the simplicity of the eigenvalues for fourth order O. D. E. 's can be found for instance in [LE-NE, Section 4]. There, it is in fact asserted that simplicity is a quite general property of fourth order equations with regular coefficients, provided they are given in "variational" form. However, the precise statement we need here with irregular coefficients and with our specific choice of boundary conditions was not given explicitly in [LE-NE]. Moreover, we need also the observability results.

We begin with an improvement of a result of [LE-NE] (Lemma 2.1). Set $E = \{v \in H^2(0,1);\ av_{xx} \in H^2(0,1)\}$, and consider the equation

$$(a\,\phi_{xx})_{xx} = \lambda\, b\, \phi \quad \text{in } (0,1)\,. \tag{A_2}$$

Lemma A_1. *Let $\phi \in E$ be a solution of Equation (A_2). Assume there exists $\xi \in [0, 1)$ such that $\phi(\xi)$, $\phi_x(\xi)$, $(a\,\phi_{xx})(\xi)$, $(a\,\phi_{xx})_x(\xi)$ are ≥ 0, and $\phi(\xi) + \phi_x(\xi) > 0$.*
Then ϕ, ϕ_x, $a\,\phi_{xx}$, and $(a\,\phi_{xx})_x$ are > 0 on $(\xi, 1]$.

Proof. We integrate Equation (A_2) from $\xi < x$ to x:

$$\begin{aligned} (a\,\phi_{xx})_x(x) &= (a\,\phi_{xx})_x(\xi) + \int_\xi^x (a\phi_{xx})_{xx}(t)\,dt \\ &= (a\,\phi_{xx})_x(\xi) + \lambda \int_\xi^x (b\,\phi)(t)\,dt\,. \end{aligned} \tag{A_3}$$

Integrating once more, we get

$$(a\,\phi_{xx})(x) - (a\,\phi_{xx})(\xi) = (x-\xi)(a\,\phi_{xx})_x(\xi) + \lambda \int_{\xi}^{x} dt \int_{\xi}^{t} (b\,\phi)(z)\,dz$$

$$= (x-\xi)(a\,\phi_{xx})_x(\xi) + \lambda \int_{\xi}^{x} (x-t)(b\,\phi)(t)\,dt \quad \Rightarrow$$

$$(a\,\phi_{xx})(x) = (a\,\phi_{xx})(\xi) + (x-\xi)(a\,\phi_{xx})_x(\xi) + \lambda \int_{\xi}^{x} (x-t)\,(b\,\phi)(t)\,dt\,. \qquad (A_4)$$

Since $\phi(\xi) + \phi_x(\xi) > 0$ and $\phi(\xi) \geq 0$, there exists $\eta > 0$ such that $\phi > 0$ on $(\xi, \eta]$. Let $\eta \leq 1$ as large as possible, and suppose $\eta < 1$, that is, $\phi(\eta) = 0$. By (A_4) and assumptions in Lemma A_1, $a\,\phi_{xx} \geq 0$ on $[\xi, \eta]$. Thus ϕ_x is nondecreasing, and therefore ≥ 0 on $[\xi, \eta]$.

Then ϕ is also nondecreasing on $[\xi, \eta]$. But this contradicts $\phi(\eta) = 0$. Thus $\eta = 1$ and ϕ is > 0 on $(\xi, 1]$. By (A_3), (A_4), and integration of (A_4), the same is true for $(a\,\phi_{xx})_x$, $(a\,\phi_{xx})$, and ϕ_x. This completes the proof. ♦

Corollary A_2. *Let $\phi \in E$ be a solution of Equation (A_2) such that $\phi(1) \geq 0$, $\phi_x(1) \leq 0$, $(a\,\phi_{xx})(1) \geq 0$, $(a\,\phi_{xx})_x(1) \leq 0$, and $\phi(1) - \phi_x(1) > 0$. Then $\phi > 0$ on $[0, 1)$.*

Proof. We set $\psi(x) = \phi(1-x)$. Then ψ satisfies

$$(\underline{a}\,\psi_{xx})_{xx} = \lambda\,\underline{b}\,\psi \quad \text{on } (0, 1) \qquad (A_5)$$

with $\psi(0) = \phi(1) \geq 0$, $\psi_x(0) = -\phi_x(1) \geq 0$, $(\underline{a}\,\psi_{xx})(0) = (a\,\phi_{xx})(1) \geq 0$, $(\underline{a}\,\psi_{xx})_x(0) = -\,(a\,\phi_{xx})_x(1) \geq 0$, $\psi(0) + \psi_x(0) > 0$, and $\underline{a}(x) = a(1-x)$, $\underline{b}(x) = b(1-x)$. Then Lemma A_1 applies to Equation (A_5). ♦

Corollary A_3. *Let $\phi \in E$ be a solution of the following problem*

$$\begin{cases} (a\,\phi_{xx})_{xx} = \lambda\,b\,\phi \quad \text{in } (0, 1) \\ \phi(0) = \phi_x(0) = 0 \\ (a\,\phi_{xx})(1) = (a\,\phi_{xx})_x(1) = 0 \\ \phi(1) \text{ or } \phi_x(1) = 0 \end{cases} \qquad (A_6)$$

Then $\phi \equiv 0$ on $(0, 1)$

Proof.

(i) Case $\phi(1) = 0$. Assume $\phi_x(1) \neq 0$, for instance $\phi_x(1) < 0$, without restriction. By Corollary A_2, $\phi > 0$ on $[0, 1)$, thus $\phi(0) > 0$, which is a contradiction. Therefore,

$$\phi(1) = \phi_x(1) = (a\,\phi_{xx})(1) = (a\,\phi_{xx})_x(1) = 0.$$

Setting $a\,\phi_{xx} = z$, (A_6) is then equivalent to the Cauchy problem in (ϕ, ϕ_x, z, z_x)

$$\begin{cases} \phi_{xx} = \dfrac{z}{a}, \quad z_{xx} = \lambda\, b\, \phi \\ \phi(1) = \phi_x(1) = 0\ ,\ z(1) = z_x(1) = 0 \end{cases}$$

thus $\phi \equiv z \equiv 0$.

(ii) Case $\phi_x(1) = 0$. Assume $\phi(1) \neq 0$, for instance $\phi(1) > 0$, without restriction. Apply Corollary A_2 to get a contradiction. Thus $\phi(1) = \phi_x(1) = 0$ and we conclude as in case (i). ♦

Proof of Proposition 3.5. Since $\phi_n \not\equiv 0$, $\phi_n(1)$ and $\phi_{nx}(1)$ are nonzero by Corollary A_3. Set $\phi_n(1) = \mu$, then $w = \dfrac{\phi_n}{\mu}$ is a solution of

$$\begin{cases} (a\, w_{xx})_{xx} = \lambda_n\, b\, w \quad \text{in } (0,\, 1) \\ w(0) = w_x(0) = 0 \\ (a\, w_{xx})(1) = (a\, w_{xx})_x(1) = 0 \\ w(1) = 1 \end{cases} \tag{A_7}$$

By Corollary A_3, (A_7) has a unique solution w. Therefore $\phi_n = \mu\, w = \phi_n(1)\, w$, so the eigenspace associated with λ_n has dimension one. This proves Proposition 3.5. ♦

REFERENCES

[BA-SLE] J. M. BALL, M. SLEMROD, Nonharmonic Fourier series and the stabilization of distributed semilinear control systems, Comm. Pure Appl. Math. 32 (1979) pp. 555-587.

[BRE1] H. BREZIS, Opérateurs Maximaux Monotones et Semi-groupes de Contraction dans les Espaces de Hilbert, North-Holland (1973).

[BRE2] H. BREZIS, Monotonicity Methods in Hilbert Spaces and Some Applications to Nonlinear Partial Differential Equations, in Contributions to Nonlinear Functional Analysis, ed. E.H. Zarantonello, Ac. Press (1971), pp 101-157.

[CHE-DE] G. CHEN, M. DELFOUR, A. M. KRALL, and G. PAYRE, Modeling, stabilization and control of serially connected beams, SIAM J. Contr. Opt. 25, 3 (1987) pp. 526-546.

[CHE-KRA] G. CHEN, S. G. KRANTZ, D. W. MA, C. E. WAYNE, and H. H. WEST, The Euler-Bernoulli beam equations with boundary energy dissipation, in Operator Methods for Optimal Control Problems, Sung J. Lee Eds., Marcel Dekker (1988) pp. 66-96.

[DA-SLE] C. M. DAFERMOS, M. SLEMROD, Asymptotic behaviour of nonlinear contraction semi-groups, J. Funct. Anal. 13 (1973) pp. 97-106.

[LA] J. E. LAGNESE, Boundary Stabilization of Thin Plates (to appear).

[LA1] I. LASIECKA, Asymptotic behaviour of the solutions to wave equation with nonlinear damping on the boundary, Proc. Conf. Anal. Opt. Systèmes, Antibes (1988) pp. 472-483.

[LA2] I. LASIECKA, Stabilization of wave and plate like equations with nonlinear dissipation on the boundary, J. Diff. Eqs. (to appear).

[LA3] I. LASIECKA, Asymptotic behavior of solutions to the plate equations with nonlinear dissipation occurring through the shear forces and bending moments, Appl. Math. Report 88-15 (1988) Univ. of Virginia, Charlottesville.

[LA4] I. LASIECKA, Asymptotic behaviour of the solutions of the Kirchoff plate with nonlinear dissipation in the bending moment, Proc. IFIP Conf. Clermont-Ferrand (june 1988), in Control of Boundaries and Stabilization, J. Simon Ed., Lect Notes in Contr. and Inf. Sci. 125 (1989) pp. 168-176.

[LE-NE] W. LEIGHTON, Z. NEHARI, On the oscillation of solutions of self-adjoint linear differential equations of the fourth order, Trans. Amer. Math. Soc. 89 (1958) pp. 325-377.

[PA] A. PAZY, Initial value problems for nonlinear differential equations in Banach spaces, Proc. Sem. Coll. de France 84, 5 (1982) pp. 154-172.

[RU] D. L. RUSSEL, Mathematical models for the elastic beams and their control theoretic implications, in Semi-groups, Theory and Applications, Eds. H. Brezis, M.G. Crandall, F. Kappel, Longman (1985) pp. 177-216.

[SLE] M. SLEMROD, Feedback stabilization of a linear control in Hilbert space with an a priori bounded control, Math. Control Signal Systems (1988) (to appear).

[WA-CHE] H. K. WANG, G. CHEN, Asymptotic behaviour of solutions of the one dimensional wave equation with a nonlinear boundary stabilizer, SIAM J. Contr. Opt. 27, 4 (1989) pp. 758-775.

[ZU] E. ZUAZUA, Uniform stabilization of the wave equation by nonlinear boundary feedback, SIAM J. Contr. Opt. (to appear).

Imprimé en France
par
l'Institut National de Recherche en Informatique et en Automatique

ISSN 0249-6399